TECHNISCHE UNIVERSITÄT DRESDEN

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Monte-Carlo integration: Stratified sampling

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Advanced Seminar for Monte-Carlo Methods

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"640 kB ought to be enough for anybody."

Bill Gates (*1955)

http://de.wikiquote.org/wiki/Bill_Gates

Review of crude Monte-Carlo integration

Goal: Result of Integral of given function f:

$$I = \int dx f(x) = \int d^d u f(u_1, ..., u_d)$$

Approximation (Monte-Carlo integration):

$$E = \frac{1}{N} \sum_{n=1}^{N} f(x_n)$$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = I$$

Review of error estimation for crude M.-C. integration

Variance for $\sigma^2(f)$:

$$\sigma^2(f) = \int dx\,(f(x) - I)^2$$

$$\int dx_1 \ldots \int dx_N \left(\frac{1}{N}\sum_{n=1}^{N} f(x_n) - I\right)^2 = \frac{\sigma^2(f)}{N}$$

Error of M-C: $\sigma(f)/\sqrt{N}$

Stratified sampling

Idea: reducing $\sigma^2(f)$

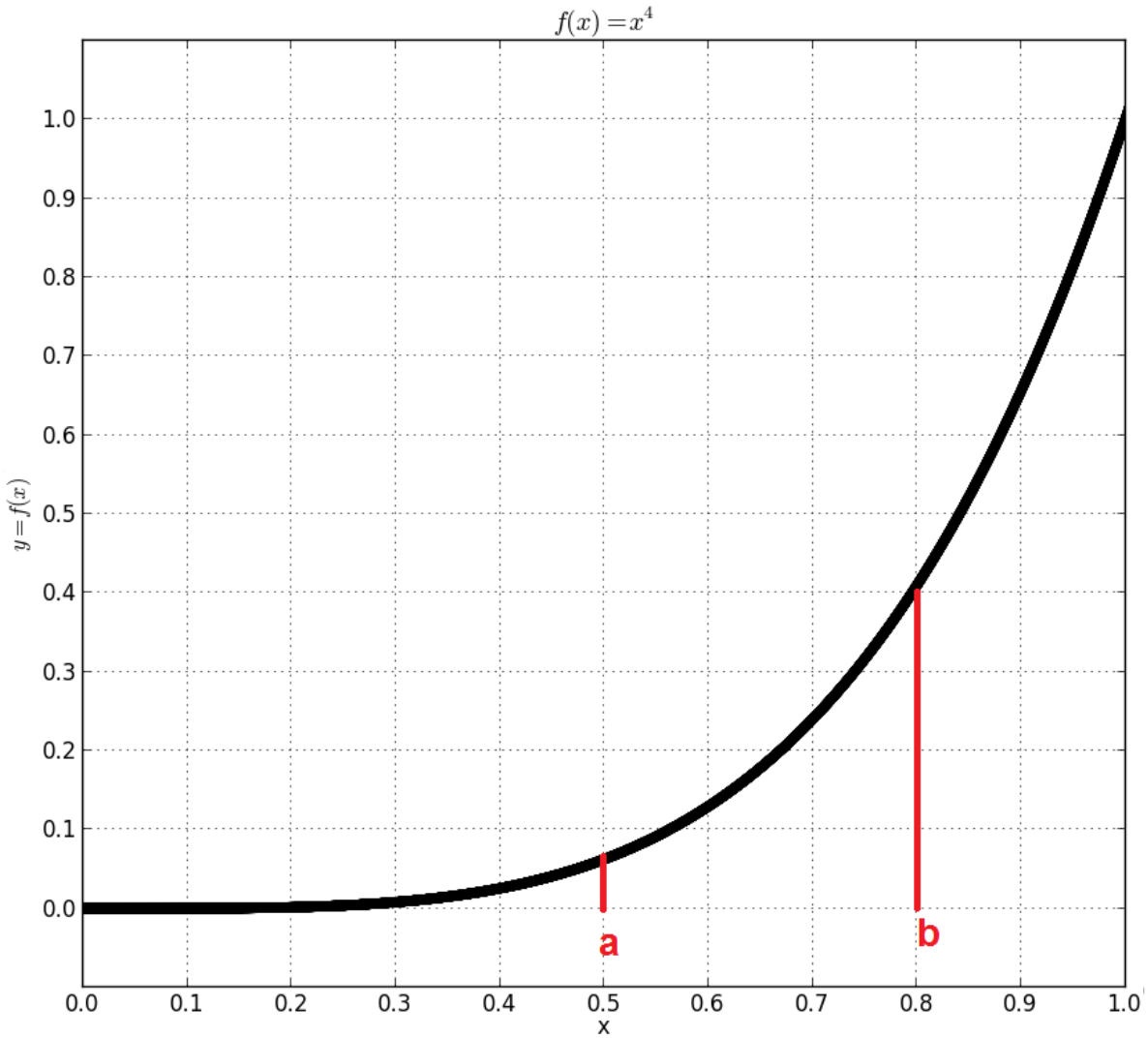
Approach:

$$\int_0^1 dx f(x) = \int_0^a dx f(x) + \int_a^1 dx f(x), \quad 0 < a < 1.$$

Result:

$$E = \sum_{j=1}^{k} \frac{\mathrm{vol}(M_j)}{N_j} \sum_{n=1}^{N_j} f(x_{jn})$$

Stratified sampling



Error estimation for stratified sampling

Error for M.-C. with s.s.:

$$\sum_{j=1}^{k} \frac{\mathrm{vol}(M_j)^2}{N_j} \, \sigma^2(f)\big|_{M_j}$$

Where:

$$\begin{aligned} \sigma^2(f)\big|_{M_j} &= \frac{1}{\mathrm{vol}(M_j)} \int_{M_j} dx \left(f(x) - \frac{1}{\mathrm{vol}(M_j)} \int_{M_j} dx \; f(x) \right)^2 \\ &= \left(\frac{1}{\mathrm{vol}(M_j)} \int_{M_j} dx \; f(x)^2 \right) - \left(\frac{1}{\mathrm{vol}(M_j)} \int_{M_j} dx \; f(x) \right)^2 \end{aligned}$$

Error estimation of stratified sampling

Choice of number of points N_j :

$$\frac{N_j}{N} = \frac{\sigma(f)|_{M_j}}{\sum_{j=1}^{N} \sigma(f)|_{M_j}}$$

$$\sigma\left(\text{Result}\right) = \sum_{j=1}^{k} \frac{\text{vol}\left(M_j\right)^2}{N_j} \sigma(f)|_{M_j}$$

Where:

$$\sigma^2\left(f\right) = \frac{1}{N} \sum_{n=1}^{N} \left(f\left(x_i\right)\right)^2 - \left(\frac{1}{N} \sum_{n=1}^{N} f\left(x_i\right)\right)^2$$

$$\sigma^2\left(f\right) = \frac{1}{4} \left(\frac{\sigma^2(f)|_a}{N_a} + \frac{\sigma^2(f)|_b}{N_b}\right) \overline{f_j} \int_{M_j} dx\, f(x)$$

Thank you for your attention!

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References

- Stefan Weinzierl: Introduction to Monte-Carlo methods

- http://de.wikiquote.org/wiki/Bill_Gates

- http://en.wikipedia.org/wiki/Monte_Carlo_integration#cite_ref-6